Product of hyperfnction

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Kobayasi Tamotu

(1) Operation at each point

Product is operation at each point. With respect to approximate fuction F(x) of hyperfunction f(x) and approximate fuction G(x) of hyperfunction g(x), function J(x) is calculated by formula(1).

$$J(x)=F(x)\cdot G(x) \qquad (1)$$

Hyperfunction j(x) is defined by function J(x) as is the approximate fnction. Components $j_h(x)$、$j_d(x)$、$j_1(x)$、$j_2(x)$、・・・ are calculated by formula(2)～formula(4)

$$j_h(x)=\lim_{\rho\to 0}\lim_{\varepsilon\to 0} J(x-\rho) \qquad (2)$$

$$j_d(x)=\lim_{\rho\to 0}\lim_{\varepsilon\to 0} \{J(x+\rho)-J(x-\rho)\} \qquad (3)$$

$$j_n(x)=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{x-\rho}^{x+\rho}(t-x)^{n-1}J(t)\,dt \qquad (n=1、2、3、\cdots) \qquad (4)$$

j(x) expressed by formula(5) separating components with comma ″.″and arraying components and wrapping in bractets, is defined as result of opperation.

$$j(x)=\{j_h(x)、j_d(x)、j_1(x)、j_2(x)、\cdots\} \qquad (5)$$

When formula(1) is substituted into formula(4), variable x is changed to variable t as formula(6).

$$J(t)=F(t)\cdot G(t) \qquad (6)$$

(2) Possibility of divergence

Both formula(2) and formula(3) converge, but sometimes formula(4) does not converge. Therefore product of 2 hyperfunctions f(x) and g(x) can not be always defined. For example, when approximate function $B_1(x)$ of formula(7) is substituted into approximate function F(x) and G(x) of formula(1), formulla(8) is obtained.

$$\boldsymbol{B}_1(x)=\frac{1}{\varepsilon\sqrt{\pi}}\exp\left(-\left(\frac{x}{\varepsilon}\right)^2\right) \qquad (7)$$

$$J(x)=\{\boldsymbol{B}_1(x)\}^2 \qquad (8)$$

If first component $j_1(0)$ at point x＝0 is calculated as formula(9), substituting function J(x) of formulla(8) into formula (4), $j_1(0)$ diverges to +∞.

$$j_1(0)=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{-\rho}^{+\rho}J(x)\,dx=+\infty \qquad (9)$$

(3) Expectation of convergence

When formula(4) is observed, if approximate function G(t) is the form of formula (10) containing variable x, it is expected that formula(4) may converge.

$$G(t)=(t-x)^k \quad (k=0、1、2、\cdots) \qquad (10)$$

When formula(4) is transformed into formula(11) using function G(t) of formula(10), $j_n(x)$ does converge.

$$\begin{aligned} j_n(x)&=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{x-\rho}^{x+\rho}(t-x)^{n-1}G(t)\cdot F(t)\,dt \\ &=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{x-\rho}^{x+\rho}(t-x)^{n-1}(t-x)^{k}\cdot F(t)\,dt \\ &=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{x-\rho}^{x+\rho}(t-x)^{n+k-1}\cdot F(t)\,dt \\ &=f_{n+k}(x) \end{aligned} \qquad (11)$$

The role of variable t and variable x should be noted. Variable t performs the role of independent variable at first, and diminishes after integral calculation is completed. Variable x performs the role of constant at first, and performs the role of independent variable after integral calculation is completed.Although function

G(t) of formula(10) containes variable t and variable x which are the same nature as formula(4), function G(t) of right hand side of formula(6) contains only variable t which is the same nature as formula(4). If function G(t) of right hand side of formula(6) is able to transformed and expressed using function G(t) of formula(10), the product does converge.

(4) Taylor expansion

When differentiable function $\phi(x)$ is performed a taylor expansion like formula (12), function $\phi(x)$ is expressed in a form of linear combination of functions G(t) of formula(10).

$$\phi(t)=\phi(x)+\frac{\phi'(x)}{1!}(t-x)+\frac{\phi''(x)}{2!}(t-x)^2+\cdots+\frac{\phi^{(k)}(x)}{k!}(t-x)^k+\cdots$$
$$=\sum_{k=0}^{\infty}\frac{\phi^{(k)}(x)}{k!}(t-x)^k \qquad (12)$$

As for k=0, $\phi^{(0)}(t)=\phi(t)$, $0!=1$ are hold. Assume that convergence radius of formula (12) is +∞. Symbol **Σ** of formula(12) expresses summation, and formula(12) is a linear combination of function G(t) of formula(10). If function $\phi(t)$ of formula(12) is substituted into approximate function G(t) of formula(6), formula(13) is obtained.

$$J(t)=\phi(t)\cdot F(t) \qquad (13)$$

When function J(t) of formula(13) is substituted into formula(4), formula(14) is obtained.

$$j_n(x)=\sum_{k=0}^{\infty}\frac{\phi^{(k)}(x)}{k!}f_{n+k}(x) \qquad (14)$$

When formula(12) and formula(14) are compared, $(t-x)^k$ of formula(12) and $f_{n+k}(x)$ of formula(14) are corresponding. Even if $(t-x)^k\to+\infty$ when $(t-x)>1$, $k\to+\infty$, formula (12) converges. So, even if $f_{n+k}(x)\to+\infty$ when $k\to+\infty$, formula(14) also converges. When function J(x) of formula(13) is substituted into formula(2), formula(15) is obtained.

$$j_h(x)=\phi(x)\cdot f_h(x) \qquad (15)$$

When function J(x) of formula(13) is substituted into formula(3), formula(16) is obtained.

$$j_d(x)=\phi(x)\cdot f_d(x) \qquad (16)$$

Among formula(14), formula(15), formula(16), only formula(15), formula(16) are the same form with formula(13). When $j_h(x)$ of formula(15), $j_d(x)$ of formula(16), $j_n(x)$ of formula(14) are substituted into formula(5), product of differentiable function $\phi(t)$ and hyperfunction f(x) is obtained.

(5) Comparison between addition and product

Product is an operation at each point. When component $j_n(x)$ is obtained assuming approximate function be product J(x) of approximate function F(x) and G(x), component $j_n(x)$ may, in some case, diverge. If component $j_n(x)$ diverges, hyperfunction j(x) is not able to be defined. Addition is also an operation at each point. When component $j_n(x)$ is obtained assuming approximate function be addition J(x) of approximate function F(x) and G(x), component $j_n(x)$ always converges. As for convergence of component, addition and product show different behavior.