Integral of hyperfunction

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(1) At a point and over an interval

Sum, multiple, sideslip, reflection, derivative, primitive of hyperfunction, product of function and hyperfunction are operations at a point, however integral is an operation over an interval from lower bound to upper bound. When operation at a point is defined, function J(x) which is the result of operation of approximate function is used as approximate function, then hyperfunction j(x) is defined. An interval $x-\rho \leqq t \leqq x+\rho$ of approximate function J(x) corresponds a point $t=x$ of hyperfunction j(x). When operation over interval is defined, operation should be converted to operation at a point, in order to correspond an interval $x-\rho \leqq t \leqq x+\rho$ of approximate function J(x) with a point $t=x$ of hyperfunction j(x), then operation can be discussed.

(2) Indefinite integral

Using approximate function F(x) of hyperfunction f(x), indefinite integral J(x) is calculated as formula(1).

$$J(x)=\int_{a-\rho}^{x} F(t)\,dt \qquad (1)$$

Function F(x) and J(x) include approximate variable ε. Variable t behaves as an independent variable during process of right hand side calculatin, and disapears when the calculation is completed. Variable x is upper bound of right hand side integral, and behaves as a constant during process of right hand side calculation, and behaves as an independent variable of left hand side function J(x) when the calculation is completed.

Hyperfunction j(x) whose aproximate function is J(x) of formula(1) is called indefinite integral of hyperfunction f(x), and is expressed as formula(2).

$$j(x)=\int_{a-0}^{x} f(t)\,dt \qquad (2)$$

Components of hyperfunction j(x)is calculated by formula(3)～formula(5).

$$j_h(x)=\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{a-\rho}^{x-\rho} F(x)\,dx \qquad (3)$$

$$j_d(x)=f_1(x) \qquad (4)$$

$$j_n(b)=-\frac{1}{n}f_{n+1}(b) \qquad (n=1、2、3、\cdots) \qquad (5)$$

If formula(3)～formula(5) are substituted into formula(6), indefinite integral j(x) is expressed by the form of function array.

$$j(x)=\{j_h(x), j_d(x), j_1(x), j_2(x), \cdots\} \qquad (6)$$

If formula(2) and formula(6) are combined , formula(7) is obtained.

$$\int_{a-0}^{x} f(t)dt=\{j_h(x), j_d(x), j_1(x), j_2(x), \cdots\} \qquad (7)$$

Point $t=x$ of hyperfnction j(x) of formula (2), corresponds microdomain $x-\rho \leqq t \leqq x+\rho$ of approximate function J(x) of formula(1), being intervened by formula(3)～formula(5). Because point $t=a$ of hyperfunction f(x) corresponds whole microdomain $a-\rho \leqq t \leqq a+\rho$, each point of microdomain of approximate function F(x) does not correspond any point of hyperfunction f(x). As end point $t=a-\rho$ is an exception, and corresponds limit $t=a-0$, lower bound $a-\rho$ of right hand side integral of formula(1) corresponds lower bound a-0 of right hand side integral of formula(2), not being intervened by formula(3)～formula(5). When indefinite integral j(x) is defined, operation over an interval is converted to operation at a point of upper bound x, then operation can be discussed.

Left continuous component j(x) can not be expressed using components $f_h(x), f_d(x), f_1(x), f_2(x), \cdots$ of hyperfunction f(x). Values of components $j_d(b)、j_1(b)、j_2(b)、\cdots$ other than left continuous component

$j_h(x)$ are determined with no relation to lower bound $x=a-0$,

(3) Incoequal integral

The left hand side $j(b)$ of formula(8), formula(9) which is obtained by substituted constant $x=b$ into hyperfunction $j(x)$ of formula(2), formula(6) is called incoequal integral.

$$j(b)=\int_{a-0}^{b} f(x)\,dx \tag{8}$$

$$j(b)=\{j_h(b), j_d(b), j_1(b), j_2(b), \cdots\} \tag{9}$$

Lower bound of formula(8) is limit a-0, upper bound is constant b, and they are not coequal.

(4) Coequal integral

When $x=b+0$ is substituted into formula (2), $j(b+0)$ is obtained, both lower and upper bound are limits, which are coequal. When limit $x=b+0$ is substituted into upper bound of formula(7), formula(10) is obtained.

$$\int_{a-0}^{b+0} f(x)\,dx=\{j_h(b)+j_d(b), 0, 0, 0, \cdots\} \tag{10}$$

If lower and upper bound are coequal, lower and upper bound can be exchanged, same as integral of ordinary function and formula (11) holds.

$$\int_{a-0}^{b+0} f(x)\,dx=-\int_{b+0}^{a-0} f(x)\,dx \tag{11}$$

Coequal integral of right hand side of formula(11), is $k(a-0)$ calculated using approximate function $K(x)$ of formula(12).

$$K(x)=\int_{b+\rho}^{x} F(t)\,dt \tag{12}$$

(5) Definite integral

When formula(10) is expressed with the form of function pseudo value, all components other than left continuous component are 0, formula(13) is obtained.

$$\int_{a-0}^{b+0} f(x)\,dx=j_h(b)+j_d(b) \tag{13}$$

Expression of formula(13) is coequal integral expressed with the form of function pseudo value, and it seems as if it was one numerical value. Coequal integral expressed with the form of function pseudo value is called definite integral. When formula (3), formula(4) are used to calculate formula(13) is changed into formula(14).

$$\begin{aligned}\int_{a-0}^{b+0} f(x)\,dx &= \lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{a-\rho}^{b-\rho} F(x)\,dx+f_1(b) \\ &= \lim_{\rho\to 0}\lim_{\varepsilon\to 0}\int_{a-\rho}^{b+\rho} F(x)\,dx \end{aligned} \tag{14}$$

(6) Partition of integral

As for coequal integral, integral interval can be partitioned similar as integral of ordinary function expressed by formula(15).

$$\int_{a-0}^{b+0} f(x)\,dx=\int_{a-0}^{c-0} f(x)\,dx+\int_{c-0}^{b+0} f(x)\,dx \tag{15}$$

As for incoequal integral, integral interval can also be partitioned similar as integral of ordinary function expressed by formula(16).

$$\int_{a-0}^{b} f(x)\,dx=\int_{a-0}^{c-0} f(x)\,dx+\int_{c-0}^{b} f(x)\,dx \tag{16}$$

First term of right hand side of formula (16) is coequal integral. More than 2 incoequal integral can not be contained in each hand side.

(7) Primitive and indefinite integral

As for formula(1), formula(17) holds

$$J'(x)=F(x) \tag{17}$$

Because hyperfunction $j'(x)$ is defined from approximate function $J'(x)$ of formula(17), relationship between hyperfunction $j'(x)$ and hyperfunction $f(x)$ is expressed by formula(18).

$$j'(x)=f(x) \tag{18}$$

Formula(18) shows that indefinite integral $j(x)$ is one of primitive function of hyperfunction $f(x)$, and connects differential and integral.