Four types of hyperfunction theory

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(1) Introduction

As for hyperfunction, generally speaking, 3 types of theories are known. Paying notice to the characteristics of hyperfunction theory framework, they may be called functional type theory, real axis step type theory, operational type theory, respectively. The author proposes component type theory, adding to these theories. Outlines of these 4 type theories are described here.

(2) Functional type theory

The theory is called theory by Schwartz, taking name from mathematician who compiled all the past studies on the theory. It also may be called functional type theory, taking name from characteristics of the framework of the theory.

A rule, by which the value f(x) is determined depending on varriable x, when a value of varriable x is determined, is called function. A rule, by which the value $\tau(\phi)$ is determined depending on function ϕ(x), when a function ϕ(x) is determined, is called functional. Suppose that function F(x) is smooth within whole area of real number. Formula(1) expresses the rule by which the value $\tau(\phi)$ of right side of formula(1) is determined, when the part without ϕ(x) of left side of formula(1) act upon function ϕ(x).

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} F(x)\phi(x)dx=\tau(\phi) \qquad (1)$$

The rule of formula(1) is functional. The integral interval of formula(1) is whole area of real number $-\infty<x<+\infty$, the integrand F(x)ϕ(x) is limitted to rapidly decreasing function, in order that the integral converges. When formula(1) holds, approximate function F(x) defines hyperfunction f(x). At singular point, hyperfunction f(x) does not have function value.

Distribution is explained as correspondence between field of distribution and distributed quantity. Because the theory by Schwartz does not mention field of distribution and distributed quantity, it seemes not to succeed explainig distribution.

(3) Real axis step type theory

The theory is called theory by Satou taking name from mathematician who got idea of the theory. It also may be called real axis step type theory, taking name from characteristic of the framework of the theory.

Hyperfunction f(x) is defined by formula(2), using complex function F(z),

$$f(x)=\lim_{\varepsilon\to 0}\{F(x+i\varepsilon)-F(x-i\varepsilon)\} \qquad (2)$$

Function F(z) is called generating function. In case hyperfunction f(x) is an ordinary function, generating function F(z) has step along x-axis of complex plain. At singular point hyperfunction f(x) does not have function value. As far as theory by Satou is concerned, it seems not be thought that a hyperfunction expresses a distribution.

(4) Operational type theory

Operational calculous has been developed as the skill of calculation to solve differential equation algebraically. A textbook written around 1953 by Mikusinski born in Poland describes in a easy style. Because operator shows property closely related to hyperfunction, operational calculous may be called operational type theory on hyperfunction.

As far as operational calculous is concerned, function and function value are strictly distinguished. Function f act on varriable x then function value f(x) is produced. Therfore function f is also operator. Constant value and constant function are strictly distinguished. As far as function f used in operational calculous is concerned, in interval $-\infty < x < 0$, function value is 0 shown as formula(3), and is right partial function.

$$f(x) = 0 \qquad (-\infty < x < 0) \qquad (3)$$

Function operator, numerical operator, transfer operator, integral operator, differential operator are defined, and product and sum are defined. Related to product and sam, distribution law holds. Product sf of differential operator s and function f have relation with derivative f' shown as formula(4).

$$f' = sf - f(0) \qquad (4)$$

If differential equation is transformed using formula(4), as product and sum are defined defferential equation can be solved by algebraical calculous. As far as operational calculous is concerned, constant function [1] and integral operator are the same. Constant function [1] is almost the same as Heviside function. Numerical operator 1 is almost the same as Dirac function. As far as operational calculous is conerned, it seems not to be thought that an operator expresses a distribution.

(5) Component type theory

A rule, by which the value f(x) is determined depending on varriable x, when a value of varriable x is determined, is called function. Because hyperfunction is understood as the expanded concept of function, the basic definition of function should not be changed. As for exisiting 3 theories, function value at singular point is not defined. In order to express function value at singular point, component expression is introduced. Suppose that approximate function F(x) containing singularizing parameter ε is smooth in the domain. Left continuous component $f_h(x)$, step component $f_d(x)$, first degree component $f_1(x)$, ・・・, n-th degree component $f_n(x)$, ・・・are calculated by formula(5)～formula(8), using approximate f unction F(x).

$$f_h(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} F(x-\rho) \qquad (5)$$

$$f_d(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \{F(x+\rho) - F(x-\rho)\} \qquad (6)$$

$$f_1(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \int_{x-\rho}^{x+\rho} F(t)\,dt \qquad (7)$$

$$f_n(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \int_{x-\rho}^{x+\rho} (t-x)^{n-1} F(t)\,dt \qquad (8)$$

Components are made a set as shown by formula(9).

$$f(x) = \{f_h(x), f_d(x), f_1(x), \cdots, f_n(x), \cdots\} \qquad (9)$$

Formula(5)～formula(8) calculate components at point t＝x of hyoerfuntion f(x) from the state of approximate function F(x) within the interval $x-\rho \leqq t \leqq x+\rho$. Interval $x-\rho \leqq t \leqq x+\rho$ is called microdomain, and parameter ρ is called microdomain radius parameter.

Domain of independent variable correspond field of distribution. and range of dependent variable correspond distributed quantity. Component type theory can explain about distribution.

(6) Conclusion

As far as existing 2 theories which are functional type theory and real axis step type theory are concerned, function value at singular point is not defined. The nature of correspondence between independent variable and dependent variable is lost at singular point. Operator except function operator of operational type theory originally dos not have the nature of correspondene between independent variable and dependent variable. Component type theory keep the nature of correspondence between independent variable and dependent variable.