Component type hyperfunction

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(1) Approximate function

When a function F(x) containing independent variable x and singularizing parameter ε is infinitely differentiable, hyperfunction f(x) can be defined using the function F(x) as the approximate function.

(2) Definition of component

Using approximate function F(x) and microdomain radius parameter ρ, component $f_h(x), f_d(x), f_1(x), \cdots, f_n(x), \cdots$ of hyperfunction f(x) are calculated by formula(1)～formula(4).

$$f_h(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} F(x-\rho) \qquad (1)$$

$$f_d(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \{F(x+\rho)-F(x-\rho)\} \qquad (2)$$

$$f_1(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \int_{x-\rho}^{x+\rho} F(t)\,dt \qquad (3)$$

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$$f_n(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \int_{x-\rho}^{x+\rho} (t-x)^{n-1} F(t)\,dt \qquad (4)$$

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Note that subscript n of left hand side and power index n-1 of right hand side is not consistent. Formula(1)～formula(4) calculate components $f_h(x), f_d(x), f_1(x), \cdots, f_n(x), \cdots$ at t＝x from state of approximate function F(x) within interval $x-\rho \leqq t \leqq x+\rho$. Interval $x-\rho \leqq t \leqq x+\rho$ concerning approximate function F(x) is called microdomain concerning hyperfunction f(x), and parameter ρ is called microdomain radius parameter. During formula(1)～formula(4) are calculating singularizing parameter ε varies faster to limit than microdomain radius parameter ρ. Components are called like this, $f_h(x)$ is left continuous component, $f_d(x)$ is step component, $f_1(x)$ is first degree component, ・・・, $f_n(x)$ is n-th degree component, ・・・. Number of components is infinite.

(3) Expression of hyperfunction

Separating with comma ″,″ lining up, putting in brackets components $f_h(x), f_d(x), f_1(x), \cdots, f_n(x), \cdots$, with the same expression as a vector, hyperfunction f(x) is expressed by formula(5).

$$f(x) = \{f_h(x), f_d(x), f_1(x), \cdots, f_n(x), \cdots\} \qquad (5)$$

Expression by formula(5) is called function array. Letting symbols 1, ⇡, ↑•→, ・・・, $(↑•→)^n$, ・・・, be the basis vector, and with the same expression as a vector, hyperfunction f(x) is expressed by formula (6).

$$f(x) = f_h(x) + f_d(x)\,⇡ + f_1(x)\,↑•→ + \cdots + f_n(x)\,(↑•→)^n + \cdots \qquad (6)$$

Expression by formula(6) is called function pseudo value. Symbol ⇡ is called step unit, symbol ↑•→ is called lateral axis unit.

(4) Domain

Domain of hyperfunction is an interval, and is expressed using real number a,b, putting independent variable x between one of a≦, a＜, -∞＜, as symbol of lower limit and one of ≦b, ＜b, ＜+∞, as symbol of lower limit. As the point t＝x of hyperfunction f(x) corresponds with the microdomain $x-\rho \leqq t \leqq x+\rho$ of approximate function F(x), the domain of approximate function F(x) is expressed containing microdomain radius parameter ρ. Symbol of lower limit a≦ of hyperfunction f(x) corresponds symbol of lower limit $a-\rho\leqq$ of approximate function F(x). Symbol of lower limit a＜ of hyperfunction f(x) corresponds symbol of lower limit $a+\rho\leqq$ of approximate function F(x). Symbol of lower limit $-\infty<$ of hyperfunction $f(x)$ corresponds symbol of lower limit $-\infty<$ of approximate function $F(x)$. Symbol of upper limit $\leqq b$ of hyperfunction $f(x)$ corresponds symbol of upper limit $\leqq b+\rho$ of approximate function $F(x)$. Symbol of upper limit $<b$ of hyperfunction $f(x)$ corresponds symbol of upper limit $\leqq b-\rho$ of approximate function $F(x)$. Symbol of upper limit $<+\infty$ of hyperfunction $f(x)$ corresponds symbol of upper limit $<+\infty$ of approximate function $F(x)$.

(5) Singular point and ordinary point

A point, where any components $f_d(x)$, $f_1(x)$, ・・・, $f_n(x)$, ・・・, other than left continuous component $f_h(x)$, is not 0, is called singular point of hyperfunction. A point which is not singular point is called ordinary point of hyperfunction. On the one hand, ordinary point exist continuously, the other, singular point exist discretely. A point, where step component $f_d(x)$ is not 0, is called step point. A point, where n-th degree component $f_n(x)$ is not 0, is called n-th degree concentration point.

(6) Equivalent approximate function

When components calculated from 2 approximate functions $F(x)$, $G(x)$ satisfy formula(7)～formula(10), functions $F(x)$ and $G(x)$ are defined as equivalent.

$$f_h(x)=g_h(x) \qquad (7)$$

$$f_d(x)=g_d(x) \qquad (8)$$

$$f_1(x)=g_1(x) \qquad (9)$$

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$$f_n(x)=g_n(x) \qquad (10)$$

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When approximate function $F(x)$ and $G(x)$ are equivalent, substituting formula(7)～formula(10) into formula(5) or formula(6), formula(11) is obtained.

$$f(x)=g(x) \qquad (11)$$

Equivalent approximate function exist in infinitely many.

(7) Not containing singularizing parameter

When approximate function $F(x)$ does not contain singularizing parameter ε, substituting into formula(1)～formula(4) and calculating, then substituting into formula(5), formula(12) expressed in function array is obtained.

$$f(x)=(F(x),0,0,0,\cdots,0,\cdots) \qquad (12)$$

Substituting into formula(6), formula(13) expressed in function pseudo value is obtained.

$$f(x)=F(x)+0\cdot\lrcorner\!\!\uparrow\!\!\ulcorner+0\cdot\uparrow_{\bullet\to}+0\cdot\uparrow_{\bullet\to}^{2}+\cdots+0\cdot\uparrow_{\bullet\to}^{n}+\cdots$$
$$=F(x) \qquad (13)$$

As far as formula(12),formula(13) are concerned, all components other than left continuous component are 0. Concerning formula(13), hyperfunction $f(x)$ seems to be apparently the same as approximate function $F(x)$.

When approximate function $G(x)$ containing singularizing parameter ε converges approximate function $F(x)$ not containing singularizing parameter ε, as formula(14), substituting approximate function $G(x)$ into formula(1)～formula(4) and calculating, then substituting into formula(5), hyperfunction $g(x)$ is expressed in function array by formula(15)

$$\lim_{\varepsilon\to 0} G(x)=F(x) \qquad (14)$$

$$g(x)=(F(x),0,0,0,\cdots,0,\cdots) \qquad (15)$$

Because right hand side of formula(12) and formula(15) coincide, approximate function $G(x)$ and approximate function $F(x)$ are equivalent. For example, approximate function $G(x)$ expressed by formula(16) is equivalent to approximate function $F(x)$ expressed by formula(17)

$$G(x)=\varepsilon x^2+(1+\varepsilon)x+3+\varepsilon^2 \qquad (16)$$

$$F(x)=x+3 \qquad (17)$$