A hyper function expresses a distribution

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The structural mechanics defines three kinds of load, that is distributed force, concentrated force, concentrated moment. Using the hyper function, the three kinds of load may be expressed in the same way. And the concentrated force is expressed by the singular point of the Dirac function. The theory of the hyper function explains that value at the singular point of the Dirac function does not exist. The author proposes that the characteristics of the singular point of the Dirac function should be understood as the concentration of the definite integration. Using this concept, something like function value is defined, then distribution of load is expressed as the correspondence between coordinate along member axis and the something like function value.

１. Introduction

A distribution can be expressed using a function. A function is a correspondence relation between values of an independent variable and values of a dependent variable. As shown in Figure-1, the distribution of deflection of a

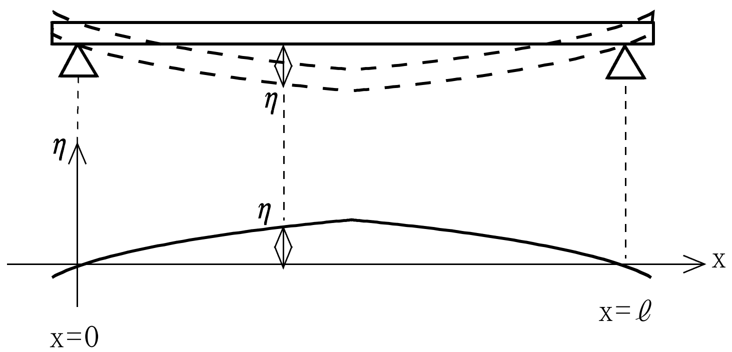

Figure-1 Distribution of Deflection

simple beam can be understood as a correspondence relation between the coordinate x along the member axis and the deflection η. In the top part of Figure-1, positive deflection is downward as is customary in structural mechanics, but in the bottom graph of Figure-1, the positive direction is upward in accordance with the mathematical convention. Since the numerical value x and the value η associated with it are in a correspondence relation, this relation is a function.

The load acting on the simple beam has three different types of units. The load distribution cannot be expressed by a function in such a way that all three units are represented in an integrated manner. In this report, we discuss hyper functions as a way of expressing load distributions.

２. How the Structural Mechanics Describes Loads

As shown in Figure-2, the load f(x) consists of a distributed force w(x) measured in N/m, concentrated force p(x) measured in N, and concentrated moment μ(x) measured in Nm. The load f(x) acting on the beam is distributed over the one dimensional space described by the coordinate x. In order to consider these

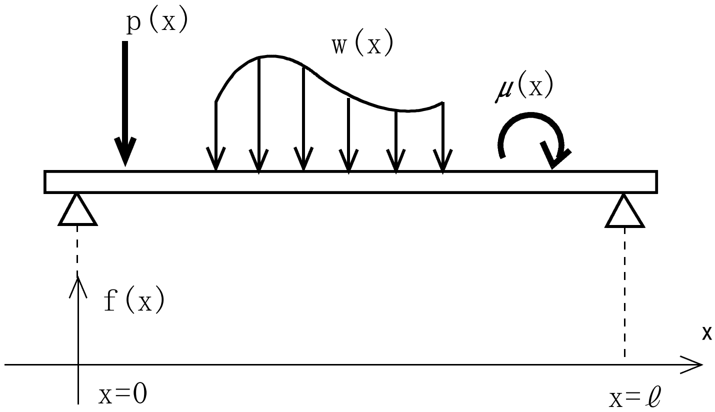

Figure-2 Load Distribution

three units simultaneously, we use N/m as the representative unit, which will be used to express the load f(x) as Formula(1).

$$f(x)\mathrm{N/m} = w(x)\mathrm{N/m} + p(x)\mathrm{N} + \mu(x)\mathrm{Nm} \quad \cdots (1)$$

By dividing this equation by the representative unit N/m, we get Formula(2).

$$f(x) = w(x) + p(x)\mathrm{m} + \mu(x)\mathrm{m}^2 \quad \cdots (2)$$

Here, ″m″ is the unit used for the coordinate x and thus is a unit of length. Let us now forget that this formula has anything to do with loads in the structural mechanics. Rather, we consider Formula(2) simply as a general, abstract mathematical formula, and refer to the unit of the horizontal axis x simply as the ″horizontal axis unit″ and denote it with the symbol ϴ. If we replace the unit ″m″ with the new symbol ϴ, we get Formula(3).

$$f(x) = w(x) + p(x)\Theta + \mu(x)\Theta^2 \quad \cdots (3)$$

The distribution of a load can be expressed using a hyper function, and the concentrated force p(x) and the concentrated moment $\mu(x)$ are expressed by points of singularity of this hyper function. However, whether one uses theory by Schwartz or theory by Sato of hyper functions, a point of singularity of a hyper function is simply described as a point where the function value does not exist. In the expression of Formula(3), the formula indicates the value of the hyper function at point x, and it is convenient to regard it as if it were a function value. We suggest that the expression of Formula(3) be called a function pseudo value. Let us now consider the concentration of a definite integral in order to discuss function pseudo values mathematically.

3. Concentration of a Definite Integral

An arbitrary integrable function f(x) satisfies the following equation at an arbitrary point x=a.

$$\lim_{\zeta \to 0} \int_{a-\zeta}^{a+\zeta} f(x)\,dx = 0 \quad \cdots (4)$$

which can be abbreviated as Formula(5).

$$\int_{a-0}^{a+0} f(x)\,dx = 0 \quad \cdots (5)$$

This is the definite integral in the interval $a-0 \leqq x \leqq a+0$. Since it is natural to identify the point x=a with the interval $a-0 \leqq x \leqq a+0$, we could just consider Formula(5) to be the definite integral at the point x=a. The definite integral of an integrable function at any point is always 0.

Now, at the singular point x=0 of the Dirac function $\delta(x)$, we get the following Formula(6).

$$\int_{-0}^{+0} \delta(x)\,dx = 1 \qquad \cdots (6)$$

This formula suggests that the definite integral in the interval $-0 \leqq x \leqq 0$ has a non zero value, contrary to Formula(5). This property is the characteristic of the singular point of the Dirac function $\delta(x)$. We suggest that this property be called the concentration of a definite integral, for the definite integral at one point is not zero.

4. Moment Definite Integral

For an arbitrary function f(x), we propose that I_k in the Formula(7) be referred to as the k-th moment definite integral at the point x=a.

$$I_k = \int_{a-0}^{a+0} (x-a)^k f(x)\,dx \qquad \cdots (7)$$

Formula(6), in particular, is the 0-th moment definite integral at the point x=0. For the derivative $\delta'(x)$ of the Dirac function, the following Formula(8) holds for its point of singularity x=0, generating the concentration of the first moment definite integral.

$$\int_{-0}^{+0} x\delta'(x)\,dx = -1 \qquad \cdots (8)$$

5. Proposal for Function Pseudo Values

The Dirac function and its derivative produce the concentration of moment definite integrals at its point of singularity, so their values could be used to describe the point of singularity. Suppose that an arbitrary function f(x) produces the concentration of the k-th moment definite integral at the point x=a. Let f(a) be the value of I_k for that function. This can be confusing unless one explicitly states that this is the concentration of the moment definite integral. If the function f(x) has a unit T, and the variable x has the unit θ, then its k-th moment definite integral has the unit $T\theta^{k+1}$. The confusion can be avoided if one writes the formula including the units, as shown below in Formula(9).

$$f(a)T = I_k T\theta^{k+1} \qquad \cdots (9)$$

Dividing through by T, we get the following Formula(10).

$$f(a) = I_k \theta^{k+1} \qquad \cdots (10)$$

In Formula(10), the symbol θ loses its meaning as a unit and becomes a symbol to indicate that the concentration of the moment definite integral is occurring. For the point of singularity of the Dirac function, we have Formula(11).

$$\delta(0) = \theta \qquad \cdots (11)$$

For the point of singularity of the derivative of the Dirac function, we have Formula(12).

$$\delta'(0) = -\theta^2 \qquad \cdots (12)$$

6. Function Pseudo Values of the Heaviside Function

The Heaviside function H(x) has a point of singularity at x=0. At this point, the function has a smooth step, and the amount of this step, H(+0)-H(-0), is the characteristic quantity of this singularity. Therefore it is natural that the quantity H(+0)-H(-0) is considered as the function pseudo value. We propose to use the symbol Θ^0 in order to indicate the existence of the smooth step. However, we should avoid conforming to the conventional definition of exponents, which would make $\Theta^0=1$, in order to avoid confusion. At the singular point of the Heaviside function, we have Formula(13).

$$H(0)=\Theta^0 \qquad \cdots (13)$$

Therefore, at the singular point x=0 of the Heaviside function H(x), we see that the step is concentrated although it is smooth.

7. Function Pseudo Values of Continuous Functions

For a continuous function f(x), we consider its value itself to be its pseudo value. For a continuous function, the value 1 is something like corresponding to each of the symbols Θ, Θ^2, Θ^0, etc.

A hyper function consists of points where it is continuous, points where the k-th moment definite integral is concentrated, and points where smooth steps are concentrated. These types of points are distinguishable by the pseudo values of the hyper function. The pseudo value can distinguish the existence or non existence of the concentration, types of the concentration by such as Θ, Θ^2, Θ^0, and 1. Function pseudo values can also be considered as vectors with these symbols as basis vectors, hyper functions then can be considered as vector valued functions.

8. Internal Structure of a Singular Point

We propose that the reason that the Dirac function $\delta(x)$ has concentration of the definite integral at its singular point is that the point has an internal structure. The overall shape of the function $\pi(x)$ expressed by Formula(15), which uses the function $\omega(x)$ given in Formula(14), is shown in Figure-3 below. Figure-3 shows the internal structure.

$$\omega(x)=\frac{(-1)^{\lambda}(2\lambda+1)!}{2^{2\lambda+1}(\lambda!)^2\varepsilon^{2\lambda+1}}(x^2-\varepsilon^2)^{\lambda} \qquad \cdots (14)$$

$$\pi(x)=\begin{cases} 0 & (-\infty<x<-\varepsilon) \\ \omega(x) & (-\varepsilon\leqq x<+\varepsilon) \\ 0 & (+\varepsilon\leqq x<+\infty) \end{cases} \qquad \cdots (15)$$

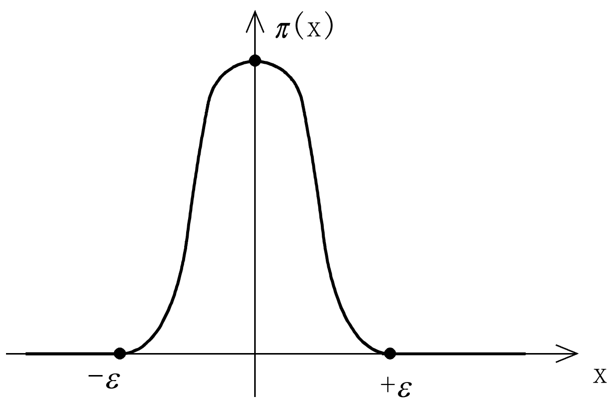

Figure-3 Internal Structure of a Singular Point

The function $\pi(x)$ is $(\lambda-1)$ times differentiable in all of the interval $-\infty<x<+\infty$, and it is infinite times differentiable if one considers the limit

$\lambda \to +\infty$. This function $\pi(x)$ also satisfies the following Formula(16).

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} \pi(x)\,dx = 1 \qquad \cdots (16)$$

Let $\varepsilon \to +0$ in this Formula(16) and compare it with Formula(6). We see that $\pi(x) \to \delta(x)$.

The function $\pi(x)$, which includes the function $\omega(x)$, shows the internal structure of the singular point x=0 of the function $\delta(x)$ in the interval $-\varepsilon \leqq x < +\varepsilon$. When we consider the limit as $\varepsilon \to +0$, the interval $-\varepsilon \leqq x < +\varepsilon$ becomes the interval $-0 \leqq x < +0$. Because this interval $-0 \leqq x < +0$ is identified as the point x=0, the internal structure described by the function $\pi(x)$ is concentrated at the point x=0.

9. Conclusion

Quantities such as a load in the structural mechanics have distributions integrating values with various units of measurement and, hence, cannot be expressed by functions. They require hyper functions. If we define a function pseudo value in order to express a hyper function, we can understand such a "distribution" as a correspondence relation between a coordinate representing the space for the distribution and a function pseudo value. The coordinate is an ordered pair of numerical values, and so is a function pseudo value, therefore, a hyper function is a function in a wider, more generalized sense of the word.

To define a function pseudo value, we focused our attention on the "concentration of a moment definite integral" among the properties of hyper functions. For the Heaviside function we focused on the "concentration of a step". We added power of "horizontal axis units" to the concentrated quantity so that we can define a function pseudo value at a point of singularity. We considered that the concentration of a definite integral occurs because the point of singularity has an "internal structure".